Midsemestral exam, Ist semester 2009
B.Math.(Hons.)Ist year
Algebra — B.Sury
Maximum Marks 60; Not all questions carry equal marks
Any score of more than 60 is equated to 60
BE BRIEF !

Q 1.
(6 marks) (i) In the group S_9, find an element σ satisfying

$$\sigma(1,3,4,9)(7,2,4)\sigma^{-1} = (8,1,6,3)(5,9,7).$$

(4 marks) (ii) Prove that if G is a group in which the map $x \mapsto x^{-1}$ is a homomorphism, then G must be abelian.

OR

(6 marks)(i) Let G be any group and $a, b \in G$. Prove that $aba^{-1}b^{-1}$ can be written as $aba^{-1}b^{-1} = x^2y^2z^2$ for some $x, y, z \in G$.
(5 marks) (ii) Let g be an element of order n in a group G and suppose $xgx^{-1} = g^d$ for some $x \in G$. Prove that $(n, d) = 1$.

Q 2.
(10 marks) If H is a subgroup of a group G such that each left coset of H is equal to some right coset, then prove that H is normal in G.

OR

(10 marks) Write out an isomorphism between the group G of eight complex matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

and the quaternion group H consisting of symbols $1, i, j, k, -1, -i - j, -k$ with the multiplication defined by

$$\pm 1.i = \pm i, \pm 1.j = \pm j, \pm 1.k = \pm k,$$

$$i^2 = j^2 = k^2 = -1, i.j = k, j.i = -k, j.k = i, k.j = -i, k.i = j, i.k = -j.$$

You may do rough work to find an isomorphism and simply write it down but do not need to show the proof why it works.

Q 3.

(12 marks) Let G be a nonabelian group of order pq where $p < q$ are primes. Prove that there is a normal subgroup of order q in G.

Hint : By Cauchy's theorem, get x, y of orders p, q respectively. If $Q := <y>$ is not normal in G, show $N_G(Q) = Q$. Use this to show $yxy^{-1} \in <x>$ and get a contradiction. Do not use Sylow's theorem which has not yet been proved for nonabelian groups.

OR

(10 marks) If $\tau_2(n)$ denotes the number of elements in S_n having order ≤ 2, prove that $\tau_2(n) = \tau_2(n-1) + (n-1)\tau_2(n-2)$ for all $n \geq 3$.

Q 4.

(10 marks) For a subgroup H of a group G, prove that the centralizer

$$C_G(H) := \{g \in G | gh = hg \forall h \in H\}$$

is a normal subgroup of the normalizer

$$N_G(H) := \{g \in G | gH = Hg\}.$$

OR

(10 marks) Let G be any group and let $\sigma : G \to G$ be an automorphism. If $\mathrm{Int}(g)$ denotes the inner automorphism $x \mapsto gxg^{-1}$ on G for any $g \in G$, show that the composite $\sigma \circ \mathrm{Int}(g) \circ \sigma^{-1} = \mathrm{Int}(\sigma(g))$.

Q 5.

(10 marks) Give an example of groups $N_1 \leq N_2 \leq N_3$ where N_1 is normal in N_2 and N_2 is normal in N_3 but N_1 is not normal in N_3.

OR

(12 marks) Show that A_4 has no subgroup of order 6.

Q 6.

(12 marks) If G is an abelian group of order p^n, where p is a prime, show that G has subgroups of each order p^r with $r \leq n$.

Hint : You may use Cauchy's theorem and induction.

OR

(10 marks) Let A be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Prove that a matrix of the form $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ where $y \in \mathbf{R}$, is expressible as $DAD^{-1}A^{-1}$ for some diagonal matrix D of the form $\mathrm{diag}(t, t^{-1})$ with $t \in \mathbf{R}^*$ if, and only if, $y \in (-1, \infty)$.

Hint : Simply compute $DAD^{-1}A^{-1}$ and see !